MATH 103B – Discussion Worksheet 3
April 20, 2023

Announcements:

1. Midterm 1 is on Wednesday April 26 during class time.
2. There will be NO discussion session on Thursday April 27.
3. In preparation of the midterm, Steve will hold office hours on Tuesday April 25 and Wednesday April 26 9:50-10:50am at AP&M 5412 *instead of* during the usual time on Thursday April 27 9:50-10:50 am.

Topic: Prime and maximal ideals (Judson 16.4)

Problem 1. Let $\varphi : \mathbb{Z}[x] \to \mathbb{C}$ be defined by $\varphi(f) = f(\sqrt{3i})$. Using the same method as in Problem 4 from last week's discussion, we can see that φ is the ring homomorphism. Compute $\operatorname{Ker}\varphi$ and $\operatorname{Im}\varphi$. Using the first isomorphism theorem, what can you conclude?

Problem 2. Consider the follow ideals in $\mathbb{R}[x]$.

1. Is (x^2-1) a prime ideal? (Hint: Recall $R[x]/(x^2-1) \cong \mathbb{R}^2$. Is $\mathbb{R}^2$ an integral domain?)
2. Is (x^2+1) a maximal ideal? (Hint: Recall $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$.)
3. Is (x^2) prime and/or maximal?

Problem 3. Is (x^2+3) prime and/or maximal in $\mathbb{Z}[x]$.

Problem 4. Let R be a commutative ring with unity. Recall the definition of nilpotent elements of R. Let $Nil(R) = \{a \in R | a \text{ nilpotent}\}$. Prove that the set of nilpotent elements of R is contained in the intersection of all prime ideals of R, i.e.

$$Nil(R) \subseteq \bigcap_{\mathfrak{p} \subseteq R \text{ prime}} \mathfrak{p}.$$

Problem 5. Let R be a commutative ring with unity, $\mathfrak{p}$ a prime ideal in R. Suppose $I_1, \ldots, I_n$ are ideals in R such that $I_1 \cdot \ldots \cdot I_n \subseteq \mathfrak{p}$. Prove that $I_j \subseteq \mathfrak{p}$ for some $j = 1, \ldots, n$.